Markov Process

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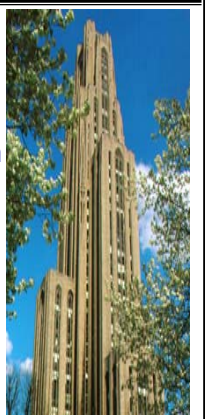


Markov Process

- In 1907 A.A. Markov defined and investigated a particular class of stochastic processes – now know as Markov processes/chains
- For a ***Markov process*** $\{X(t), t \in T\}$, *with state space S,* its future probabilistic development is dependent only on the current state, how the process arrives at the current position is irrelevant.
- Memoryless property - The process starts afresh at the time of observation and has no memory of the past.
- Previously looked at Discrete Time Markov Chain (DTMC)
- Continuous Time Markov Chain (CTMC) – often called Markov Process

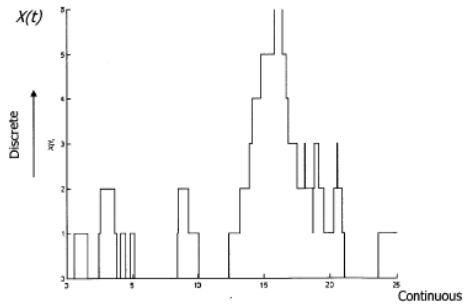
Markov Process

- The continuous time discrete state stochastic process $\{X(t), t \in T\}$ with state space S is a Markov Process or CTMC if the Markov property holds for all states and all time

  $P[\, X(t+s) = j \mid X(t) = i, X(u) = g(u),\ 0 < u < t \,]$

  $= P[X(t+s) = j \mid X(t) = i \,]$
- This is just the continuous version of the memoryless property
- The state space is discrete but now the state transitions *can occur at any point in time*.

•**Example**
X(t): Number of packets in the system at time t at a router output port



Markov Process

- The state probabilities are defined as

  $\pi_j(t) = P\{X(t) = j\}$
- The state transition probabilities are defined as

  $p_{ij}(t,s) = P\{X(t+s) = j \mid X(t) = i\}$

  if the state transitions are independent of time t the Markov Process is time homogenous and

  $p_{ij}(t) = P\{X(t) = j \mid X(0) = i\}$, $P(t) = [p_{ij}(t)]$

  the state transitions probabilities can be shown to follow

$$p_{ij}(t+s) = \sum_{k \in S} p_{ik}(t)\, p_{kj}(s)$$

Markov Process

- A time homogeneous Markov Process is characterized by the generator matrix Q = [q_{ij}] where

  q_{ij} = flow rate from state *i* to *j*

  q_{jj} = - rate of which leave state *j*

  Rows of Q sum to 0 ➔ Qe = 0

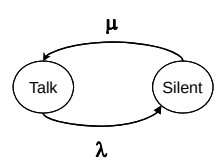
  The generator matrix can be related to the state transition probabilities as

$$\frac{dP(t)}{dt} = P(t)Q \quad \Rightarrow \quad P(t) = e^{Qt}P(0)$$

Example

Consider a two state Markov Process model of a speaker
Ref: H. Heffes and D. Lucantoni, IEEE JSAC, Sept., 1986

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$



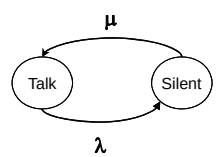
$$P(t) = e^{Qt}P(0)$$

Let P(0) = I

$$e^{Qt} = L^{-1}\{[sI-Q]^{-1}\} = L^{-1}\begin{bmatrix} \frac{s+\mu}{s(s+\lambda+\mu)} & \frac{\lambda}{s(s+\lambda+\mu)} \\ \frac{\mu}{s(s+\lambda+\mu)} & \frac{s+\lambda}{s(s+\lambda+\mu)} \end{bmatrix}$$

Example

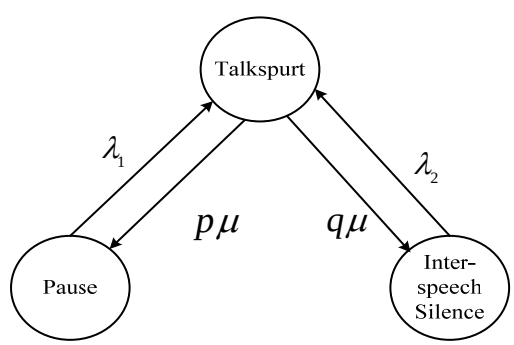
$$P(t)=e^{Qt}I=\begin{bmatrix} e^{-(\lambda+\mu)t}+\dfrac{\mu}{(\lambda+\mu)}(1-e^{-(\lambda+\mu)t}) & \dfrac{\lambda(1-e^{-(\lambda+\mu)t})}{(\lambda+\mu)} \\ \dfrac{\mu(1-e^{-(\lambda+\mu)t})}{(\lambda+\mu)} & e^{-(\lambda+\mu)t}+\dfrac{\lambda}{(\lambda+\mu)}(1-e^{-(\lambda+\mu)t}) \end{bmatrix}$$



Markov Process Example

A precise model of voice sources includes the fact that speech typically has short pause periods as well as inter-speech silence periods. Here each are assumed exponentially distributed with parameters μ , λ_1 and λ_2 respectively.

The state transition diagram of the Markov process model of a single voice source is shown below. In the diagram, p represents the probability of that a transition from the talkspurt state goes to the pause state and $q = (1 - p)$



Markov Process Example

The generator matrix Q of the Markov Process is given below : T : talkspurt, p: pause, s: interspeech silence

$$Q = \begin{array}{c} \\ \mathrm{T} \\ \mathrm{p} \\ \mathrm{s} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{T} & \mathrm{p} & \mathrm{s} \end{array} \\ \begin{bmatrix} -\mu & p\mu & q\mu \\ \lambda_1 & -\lambda_1 & 0 \\ \lambda_2 & 0 & -\lambda_2 \end{bmatrix} \end{array}$$

Determine the steady state probabilities from $\pi Q = 0$ and $\pi e = 1$

1) $-\mu\pi_T + \lambda_1\pi_p + \lambda_2\pi_s = 0$

2) $p\mu\pi_T - \lambda_1\pi_p = 0 \;\Rightarrow\; \pi_p = \dfrac{p\mu}{\lambda_1}\pi_T$

3) $q\mu\pi_T - \lambda_2\pi_s = 0 \;\Rightarrow\; \pi_s = \dfrac{q\mu}{\lambda_2}\pi_T$

4) $\pi e = 1 \;\Rightarrow\; \pi_T + \pi_p + \pi_s = 1$

Markov Process Example

Substituting from 2) and 3) into 4) yields

$$\pi_T\left(1 + \frac{p\mu}{\lambda_1} + \frac{q\mu}{\lambda_2}\right) = 1$$

$$\pi_T\left(\frac{\lambda_1\lambda_2 + p\mu\lambda_2 + q\mu\lambda_1}{\lambda_1\lambda_2}\right) = 1 \;\Rightarrow\; \pi_T = \frac{\lambda_1\lambda_2}{\lambda_1\lambda_2 + p\mu\lambda_2 + q\mu\lambda_1}$$

from 2) and 3)

$$\pi = \left[\pi_T \;,\; \pi_p \;,\; \pi_s\right] = \frac{1}{\lambda_1\lambda_2 + p\mu\lambda_2 + q\mu\lambda_1}\left[\lambda_1\lambda_2 \;,\; p\mu\lambda_2 \;,\; q\mu\lambda_1\right]$$

Markov Process Example

Typical values for the model parameters as determined from measured speech are : $\frac{1}{\lambda_1} = 105.6 \text{ msec}$ $\frac{1}{\lambda_2} = 2.712 \text{ sec}$ $\frac{1}{\mu} = 284 \text{ msec}$

$p = 0.8563$, and $q = 0.1437$

$(\lambda_1 = 9.4697,\ \lambda_2 = 0.3687,\ \mu = 3.5211)$

When the model is in the talkspurt state digitized speech is created a fixed rate of 64 kbps. Using these values determine the mean rate of the voice source.

Plugging into π on the previous slide yields

$$\pi = [0.3717,\ 0.1183,\ 0.51]$$

Mean rate $= 64\ kbps \times \pi_T = 23.7862\ kbps$

Markov Process Analysis

- The state holding time – that is the amount of time the Markov Process spends in state *i* is exponentially distributed with parameter q_{ii}. Specifically, let H_i denote the holding time in state *i* then

$$P\{H_i \le \tau\} = 1 - e^{q_{ii}\tau}$$

Markov Process Analysis

- Basic Analysis results for Markov Process

1. Behavior of state transition probabilities

$$\frac{dP(t)}{dt} = P(t)Q \quad \Rightarrow \quad P(t) = e^{Qt}P(0)$$

2. Behavior of state probabilities

$$\frac{d\pi(t)}{dt} = \pi(t)Q \quad \Rightarrow \quad \pi(t) = \pi(0)e^{Qt}$$

3. Steady state values of state probabilities

$$\pi(t) = \lim_{t\to\infty}\pi(t) \qquad \pi Q = 0 \qquad and \qquad 1 = \sum_{i\in S}\pi_i$$

4. State Holding Time

$$P\{H_i \le \tau\} = 1 - e^{q_{ii}\tau}$$