

## Time Varying Behavior

- Teletraffic typically has large time of day variations


Mean number of calls per minute at a central office switch - measured in 15 minute intervals averaged over 10 work days


Source: ITU Teletraffic Handbook

## Time Varying Behavior

- However queueing results thus far are for steady state
- Focus on steady state probabilities $\pi_{i}=\lim _{t \rightarrow \infty} P\{n(t)=i\}$
- Steady state mean behavior L, W, etc.
- What about behavior as a function of time?
- Transient:
- System going from one stationary state to another
- Nonstationary:
- System with continuous variation in arrival and/or service rates
- When does time varying/transient behavior matter?
- If load is dynamic in comparison to queue settling times
- If time varying service rate from resources being switched on and off, dynamic bandwidth allocation, etc. Service rate must change as rapid as queue
- After failure conditions


## Approximation Approaches

- Simple Stationary Approximation (SSA)
- ignore variations in load/service rates
- use average values in steady state queueing model
- simple and applicable to a wide range of queueing systems
- Good for small systems with low variation
- Peak Approximation (PA)
- use peak/maximum value instead of the average load
- widely used approach in telecom
- Quasi-Static Approximation (QSA)
- Monitor time varying parameters over a set of time intervals
- Assume static conditions during each time interval and apply steady state results for each period using mean of parameters in each period


Pointwise Stationary Approximation (PSA)

- Use sampled values of time varying parameters to evaluate steady state at each sampled timepoint


## Example

- Consider M/M/1 with $\mu=2, \lambda=1+.5 \sin (2 \pi t)$
- Focus on mean number in system $L=\rho /(1-\rho)$

|  | Mean number in System L |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
| Method | Time (0.25) | Time(.25,.5) | Time(.5,.75) | Time(.75, 1) |
| SSA <br> $\mu=2, \lambda=1$ | 1 | 1 | 1 | 1 |
| PA <br> $\mu=2, \lambda=1.5$ | 3 | 3 | 3 | 3 |
| QSA <br> $\mu=2, \lambda=1.25$, <br> $1.25, .75, .75$ | 1.667 | 1.667 | .666 | .6666 |
| PSA <br> $\mu=2, \lambda=1.35$, <br> $1.35, .646, .646$ | 2.0938 <br> $\mathrm{~T}=.125$ | 2.0938 <br> $\mathrm{~T}=.375$ | $\mathrm{~T}=.625$ | $\mathrm{~T}=.875$ |

## Steady State Behavior

- Remember basic approach is to solve system of equations derived from Markov Process model of queue together with normalization condition
- Example: Erlang B queueing model - M/M/C/C queue
- C identical servers process customers in parallel.
- Customers arrive according to a Poisson process with mean rate $\lambda$ that is independent of time
- Customer service times exponentially distributed with mean rate $\mu$ that is independent of time
- The system has a finite capacity of size $C$, customers arriving when all servers busy are dropped $\boldsymbol{\rightarrow}$ Blocked calls cleared model (BCC)



## M/M/C/C Steady State

- Analyze using Markov Process of $n(t)$ - number of customers in the system at time $t$
- Let $\pi_{i}$ be the steady state probability of $i$ customers in the system, then the state transition diagram and flow balance equations are given below

flow out state $j=$ flow in state $j$

$$
\begin{array}{rlrl}
\lambda \pi_{0} & =\mu \pi_{1} & j & =0 \\
(\lambda+j \mu) \pi_{j} & =\lambda \pi_{j-1}+(j+1) \mu \pi_{j+1} & 1 \leq j & <C \\
(C \mu) \pi_{c} & =\lambda \pi_{c-1} & j=C
\end{array}
$$

Normalization condition $\quad \sum_{j=0}^{C} \pi_{j}=1$

## M/M/C/C

Solving the equations for $\pi_{i}$, note that the basic equations are the same as for the $M / M / C$ with $j<C$. Following the analysis in previous slide set

$$
\pi_{i}=\frac{a^{i}}{i!} \pi_{0} \quad 1 \leq i \leq C
$$

Plugging into the normalization condition $\quad \sum_{j=0}^{C} \pi_{j}=1$

One gets
$\pi_{0}=\frac{1}{\sum_{n=0}^{c} \frac{a^{n}}{n!}} \Rightarrow \pi_{i}=\frac{a^{i}}{i!} \pi_{0}=\frac{\frac{a^{i}}{i!}}{\sum_{n=0}^{c} \frac{a^{n}}{n!}} \quad \forall i=1,2, \ldots c$

## Erlang B Formula

Basic QoS metric is probability of a customer being blocked $B(c, a)$

$$
B(c, a)=\pi_{c}=\frac{\frac{a^{c}}{c!}}{\sum_{n=0}^{c} \frac{a^{n}}{n!}} \quad \Leftarrow \text { Valid for M/G/c/c queue }
$$

$B(c, a) \Leftarrow$ Erlang's $B$ formula
Erlang's blocking formula
Erlangs first formula
In the telephone system,
$B(c, a)$ represents a blocked call cleared (BCC) model.

## Time Varying Behavior

- Very limited set of exact results for time varying analysis
- Basic approach is to study system of differential equations derived from Markov Process model of queue
- Example: Erlang B queueing model - M/M/C/C queue
- C identical servers process customers in parallel.
- Customers arrive according to a Poisson process with mean rate $\lambda(\mathrm{t})$ that is a function of time
- Customer service times exponentially distributed with mean rate $\mu(\mathrm{t})$ that is a function of time
- The system has a finite capacity of size $C$, customers arriving when all servers busy are dropped $\rightarrow$ Blocked calls cleared model (BCC)



## M/M/C/C/ Time Varying Model

Let $p_{i}(t)$ denote the state probability of $i$ customers in the system, from the state transition diagram for $n(t)$


Rate of change of probability of being in state $\mathrm{j}=-$ flow out state $j+$ flow in state $j$

$$
\begin{array}{lr}
d p_{0}(t) / d t=-\lambda(t) p_{0}(t)+\mu(t) p_{1}(t) & j=0 \\
d p_{j}(t) / d t=\lambda(t) p_{j-1}(t)-(\lambda(t)+j \mu(t)) p_{j}(t)+(j+1) \mu(t) p_{j+1}(t) & 1 \leq j<C \\
d p_{C}(t) / d t=\lambda(t) p_{c-1}(t)-C \mu(t) p_{c}(t) & j=C
\end{array}
$$

- The Chapman-Kolmogorov differential equation model
- Note, if set left hand side to zero get steady state flow balance equations and can solve for steady state results


## Time Varying Model

- Closed form analytical solution of C-K model not possible due to time varying coefficients
- Can be solved numerically to determine state probabilities vs. time using a standard numerical integration technique like Runge-Kutta
- Numerical solution technique can be written in algorithmic form over $\left[t_{0}, t_{f}\right]$

1) Initialization: set current time $t$, to $t=t_{0}$ establish the initial state probabilities $p\left(t_{0}\right)=\left[p_{i}\left(t_{0}\right)\right.$, $\mathrm{i}=0,1, \ldots \mathrm{C}]$ and specify a time step $\Delta t$
2) Approximate the arrival rate $\lambda(t)$ by a constant $\lambda$ over $[t, t+\Delta t]$ with $\lambda=\lambda(t+\Delta t / 2)$ and $\mu(t)$ by a constant $\mu$ over $[t, t+\Delta t]$ with $\mu=\mu(t+\Delta t / 2)$
3) Numerically solve the system of differential equations over the small time interval $\Delta t$, and get the new system state probabilities $p(t)$ at time $t+\Delta t ; p(t+\Delta t)$.
4 ) Increment time, $t=t+\Delta t$. If $t<t_{f}$, go to 2 , else stop.

- Note, number of equations grows with systems capacity C ( C > 1000 in an optical network link)
- Will be difficult to study networks of links
- Need an accurate approximation


## Fluid Flow modeling

- Consider a single transmission link

- $\mathrm{f}_{\text {in }}(\mathrm{t})=$ flow in to the queueing systems
- $x(t)=$ mean number of customers at queue at time $t$
- $f_{\text {out }}(t)=$ flow out of queueing system

$$
\dot{x}=-f_{\text {out }}(t)+f_{\text {in }}(t)
$$

Expression for flow in and flow out will depend on system under study (e.g., M/M/1, M/G/1, etc.)
Can approximate flow in/flow out by matching equilibrium point of fluid model with equivalent queueing model steady state result
See W. Wang, et.al., IEEE Infoccom 95

## Fluid Flow model

- For M/M/C/C queue

$$
\begin{gather*}
\dot{x}(t)=-f_{\text {out }}(t)+f_{\text {in }}(t) \\
f_{\text {in }}(t)=\lambda(t)\left(1-p_{C}(t)\right) \\
f_{\text {out }}(t)=\mu p_{1}(t)+2 \mu p_{2}(t)+\ldots C \mu p_{C}(t)=\mu x(t) \\
\dot{x}(t)=-\mu x(t)+\lambda(t)\left(1-p_{C}(t)\right) \tag{1}
\end{gather*}
$$

How to find $p_{c}(t)$ ?
Match Steady state results $\rightarrow$ Pointwise Stationary Fluid Flow Approximation (PSFFA)


## Fluid Flow Model

At steady state $\mathrm{dx} / \mathrm{dt}=0$ and probability of a customer being blocked $\mathrm{p}_{\mathrm{c}}(\mathrm{t})=B(\mathrm{c}, \mathrm{a}) \leftarrow$ Erlang $B$ Model

$$
\begin{align*}
& 0=-\mu x(t)+\lambda(t)\left(1-p_{C}(t)\right) \Rightarrow a(t)=\frac{x(t)}{1-p_{c}(t)}  \tag{2}\\
& B(c, a)=p_{c}=\frac{\frac{a^{c}}{c!}}{\sum_{n=0}^{c} \frac{a^{n}}{n!}} \tag{3}
\end{align*}
$$

Fixed point problem only one value of $a$ and $p_{C}(t)$ will work - solve iteratively until converges or until change in $a$ in two iteration $<\epsilon$
-Can numerically solve fluid model (1) together with fixed point equations (2) and (3) to study queue behavior

## Fluid Model Solution

Numerical solution technique can algorithmic form over $\left[t_{0}, t_{f}\right]$

1) Initialization: set current time $t$, to $t=t_{0}$ establish the initial system occupancy $x(t)=x\left(t_{0}\right)$, and specify a time step $\Delta t$
2) Approximate the arrival rate $\lambda(t)$ by a constant $\lambda$ over $[t, t+\Delta t]$ with $\lambda=$ $\lambda(t+\Delta t / 2)$
and $\mu(t)$ by a constant $\mu$ over $[t, t+\Delta t]$ with $\mu=\mu(t+\Delta t / 2)$
3) Approximate $p_{C}(t)$ over $[t, t+\Delta t]$ by a constant $p_{C}$ by solving (2) and (3) iteratively until the change in an iteration $a(x(t))$ does not exceed a prespecified $\epsilon$ value.
4) Utilizing $x(t), \lambda$ and $\mu$ (from step 2), $p_{C}$ (from step 3), numerically solve the differential equation given by (1) over the small time interval $\Delta t$, and get the new system occupancy at time $t+\Delta t ; x(t+\Delta t)$.
5) Increment time, $t=t+\Delta t$. If $t<t_{f}$, go to 2 , else stop.

## Flow Chart of Solution method



Increment time $t=t+\Delta t$

Set current time $t=t_{0}$
Initial value $x(t)=x\left(t_{0}\right)$ and
specify time step $\Delta t$

Over small interval
$\lambda=\lambda(t+\Delta t / 2) \quad \mu=\mu(t+\Delta t / 2)$

Determine $p_{C}$

Using $\mathrm{x}(\mathrm{t}), \lambda, \mu, p_{C}$
Solve for $x(t+\Delta t)$

## Numerical Results

Check the accuracy of fluid flow model vs. exact Chapman-Kolmogrov model Numerically integrate exact model - compare results with fluid flow model Results shown for $\mathrm{C}=24$ (e.g., T 1 link) $\lambda(\mathrm{t})=15+3 \sin (0.1(\mathrm{t}+20)$



## Fluid Flow Model Model

- In general for infinite capacity queues

- $\mathrm{f}_{\text {in }}(\mathrm{t})=$ flow in to the queueing systems
- $x(t)=$ mean number of customers at queue at time $t$
- $f_{\text {out }}(t)=$ flow out of queueing system

$$
\dot{x}=-f_{\text {out }}(t)+f_{\text {in }}(t)
$$

- For infinite buffer queues : $f_{\text {in }}(t)=\lambda(t), f_{\text {out }}(t)=\mu \rho(t)$ then

$$
\dot{x}=-\mu \rho(t)+\lambda(t)
$$

at steady state have $\mathrm{dx} / \mathrm{dt}=0$ and $x=G_{1}(\rho)$
Assuming $G_{1}($.$) is numerically invertible \Rightarrow \rho=G_{1}^{-1}(x)$ get

$$
\dot{x}=-\mu G_{1}^{-1}(x(t))+\lambda(t)
$$

## Fluid Flow Model Model

- Consider M/G/1 queue at steady state

$$
x=\rho+\frac{\rho\left(1+C_{S}^{2}\right)}{2(1-\rho)} \rightarrow \quad \rho=\frac{x+1-\sqrt{x^{2}+2 C_{S}^{2} x+1}}{1-C_{S}^{2}}
$$

which yields

$$
\begin{gathered}
\dot{x}=-\mu\left[\frac{x+1-\sqrt{x^{2}+2 C_{s}^{2} x+1}}{1-C_{s}^{2}}\right]+\lambda \\
\dot{x}=-\mu G_{1}^{-1}(x(t))+\lambda(t)
\end{gathered}
$$

M/G/1 Model
$\dot{x}=-\mu\left[\frac{x+1-\sqrt{x^{2}+2 C_{S}^{2} x+1}}{1-C_{S}^{2}}\right]+\lambda$

| $\mathrm{M} / \mathrm{D} / 1$ | $\dot{x}=-\mu\left[(x+1)-\sqrt{x^{2}+1}\right]+\lambda$ |
| :--- | :--- |
| $\mathrm{M} / E_{k} / 1$ | $\dot{x}=-\mu\left[\frac{k(x+1)}{k-1}-\frac{\sqrt{k^{2} x^{2}+2 k x+k^{2}}}{k-1}\right]+\lambda$ |
| M/M/1 | $\dot{x}=-\mu\left(\frac{x}{x+1}\right)+\lambda$ |

Table 1: M/G/1 PSFFA Models


## Time Varying Queueing Models

- Many other queueing models in the literature for time varying behavior - focus on numerical solution not closed form results
> Multiple traffic classes
> General Service times
> General arrival process
> Network results for simple Jackson type networks

