

IS 2150 / TEL 2810

Information Security & Privacy

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A Quick Review of Some Maths
For IS 2150/TEL 2810



Objective

- Review some mathematical concepts
 - Propositional logic
 - Predicate logic
 - Mathematical induction
 - Lattice



Propositional logic/calculus

- Atomic, declarative statements (propositions)
 - that can be shown to be either TRUE or FALSE but not both; E.g., “Sky is blue”; “3 is less than 4”
- Propositions can be composed into compound sentences using connectives
 - Negation $\neg p$ (NOT) highest precedence
 - Disjunction $p \vee q$ (OR) second precedence
 - Conjunction $p \wedge q$ (AND) second precedence
 - Implication $p \rightarrow q$ q logical consequence of p
- Exercise: Truth tables?



Propositional logic/calculus

- Contradiction:
 - Formula that is always false : $p \wedge \neg p$
 - What about: $\neg(p \wedge \neg p)$?
- Tautology:
 - Formula that is always True : $p \vee \neg p$
 - What about: $\neg(p \vee \neg p)$?
- Others
 - Exclusive OR: $p \oplus q$; p or q but not both
 - Bi-condition: $p \leftrightarrow q$ [*p if and only if q* (p iff q)]
 - Logical equivalence: $p \Leftrightarrow q$ [p is logically equivalent to q]
- Some exercises...



Some Laws of Logic

- Double negation
- DeMorgan's law
 - $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$
 - $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
- Commutative
 - $(p \vee q) \Leftrightarrow (q \vee p)$
- Associative law
 - $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
- Distributive law
 - $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
 - $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$



Predicate/first order logic

- Propositional logic
- Variable, quantifiers, constants and functions
- Consider sentence: *Every directory contains some files*
- Need to capture “every” “some”
 - $F(x)$: x is a file
 - $D(y)$: y is a directory
 - $C(x, y)$: x is a file in directory y



Predicate/first order logic

- Existential quantifiers \exists (There exists)
 - E.g., $\exists x$ is read as There exists x
- Universal quantifiers \forall (For all)
- $\forall y D(y) \rightarrow (\exists x (F(x) \wedge C(x, y)))$
- read as
 - for every y , *if* y is a directory, *then* there exists a x such that x *is a file* and x *is in directory* y
- What about $\forall x F(x) \rightarrow (\exists y (D(y) \wedge C(x, y)))$?



Mathematical Induction

- Proof technique - to prove some mathematical property
 - E.g. want to prove that $M(n)$ holds for all natural numbers
- **Base case OR Basis:**
 - Prove that $M(1)$ holds
- **Induction Hypothesis:**
 - Assert that $M(n)$ holds for $n = 1, \dots, k$
- **Induction Step:**
 - Prove that if $M(k)$ holds then $M(k+1)$ holds



Mathematical Induction

- Exercise: prove that sum of first n natural numbers is
 - $S(n): 1 + \dots + n = n(n + 1)/2$

- Prove
 - $S(n): 1^2 + \dots + n^2 = n(n + 1)(2n + 1)/6$



Lattice

- Sets
 - Collection of unique elements
 - Let S, T be sets
 - Cartesian product: $S \times T = \{(a, b) \mid a \in A, b \in B\}$
 - A set of order pairs
- Binary relation R from S to T is a subset of $S \times T$
- Binary relation R on S is a subset of $S \times S$



Lattice

- If $(a, b) \in R$ we write aRb
 - Example:
 - R is “less than equal to” (\leq)
 - For $S = \{1, 2, 3\}$
 - Example of R on S is $\{(1, 1), (1, 2), (1, 3), \text{????}\}$
 - $(1, 2) \in R$ is another way of writing $1 \leq 2$



Lattice

- Properties of relations
 - Reflexive:
 - if aRa for all $a \in S$
 - Anti-symmetric:
 - if aRb and bRa implies $a = b$ for all $a, b \in S$
 - Transitive:
 - if aRb and bRc imply that aRc for all $a, b, c \in S$
 - Which properties hold for “less than equal to” (\leq)?
 - Draw the Hasse diagram
 - Captures all the relations



Lattice

- Total ordering:
 - when the relation orders all elements
 - E.g., “less than equal to” (\leq) on natural numbers
- Partial ordering (poset):
 - the relation orders only some elements not all
 - E.g. “less than equal to” (\leq) on complex numbers; Consider $(2 + 4i)$ and $(3 + 2i)$



Lattice

- Upper bound ($u, a, b \in S$)
 - u is an upper bound of a and b means aRu and bRu
 - Least upper bound : $\text{lub}(a, b)$ *closest upper bound*
- Lower bound ($l, a, b \in S$)
 - l is a lower bound of a and b means lRa and lRb
 - Greatest lower bound : $\text{glb}(a, b)$ *closest lower bound*



Lattice

- A lattice is the combination of a set of elements S and a relation R meeting the following criteria
 - R is reflexive, antisymmetric, and transitive on the elements of S
 - For every $s, t \in S$, there exists a **greatest lower bound**
 - For every $s, t \in S$, there exists a **lowest upper bound**
- Some examples
 - $S = \{1, 2, 3\}$ and $R = \leq?$
 - $S = \{2+4i; 1+2i; 3+2i, 3+4i\}$ and $R = \leq?$



Overview of Lattice Based Models

- Confidentiality
 - Bell LaPadula Model
 - First rigorously developed model for high assurance - for military
 - Objects are classified
 - Objects may belong to Compartments
 - Subjects are given clearance
 - Classification/clearance levels form a lattice
 - Two rules
 - No read-up
 - No write-down



Summary

- Basics of Logic
- Induction technique
- Lattice