IS 2610: Data Structures

Graph

April 5, 2004

Graph Terminology

- Graph : vertices + edges
- Induced subgraph of a subset of vertices
- Connected graph: a path between every pair
  - Maximal connected: there is no path from this subgraph to an outside vertex

Representation

- Adjacency matrix
  - V by v array
  - Adv./disadvantages
- Adjacency list
  - Linked list for each vertex
  - Adv./disadvantages

```
typedef struct { int v; int w; } Edge;
Edge EDGE(int, int);

typedef  struct graph *Graph;
Graph    GRAPHinit(int);
 void      GRAPHinsertE(Graph, Edge);
 void      GRAPHremoveE(Graph, Edge);
  int        GRAPHedges(Edge [], Graph G);
Graph    GRAPHcopy(Graph);
 void      GRAPHdestroy(Graph);
```

```
typedef struct node *link;
struct node { int v; link next; };
struct graph { int V; int E; link *adj; };
Graph GRAPHinit(int V)
  { int v;
    Graph G = malloc(sizeof *G);
    G->V = V; G->E = 0;
    G->adj = malloc(V*sizeof(link));
    for (v = 0; v < V; v++) G->adj[v] = NULL;
    return G;
  }
```

Hamilton Path

- Hamilton path:
  - Given two vertices, is there a simple path connecting them that visits every vertex in the graph exactly once?
- Worst case for finding Hamilton tour is exponential
  - Assume one vertex isolated; and all v-1 vertices are connected
  - (v-1)! Edges need to be checked

Euler Tour/Path

- Euler Path
  - Is there a path connecting two vertices that uses each edge in the graph exactly once?
    - Vertices may be visited multiple times
- Euler tour: Is there a cycle with each edges exactly once
  - Bridges of konigsberg
- Properties:
  - A graph has a Euler tour iff it is connected and all the vertices are of even degree
  - A graph has a Euler path iff it is connected and exactly two of its vertices are of odd degrees
- Complexity?

Graph Search

- Depth First Search

```
void dfsR(Graph G, Edge e)
  { link t; int w = e.w;
    pre[w] = cnt++;
    for (t = G->adj[w]; t != NULL; t = t-
>next)
      if (pre[t->v] == -1)
        dfsR(G, EDGE(w, t->v));
  }
```

  - V^2 for adj matrix
  - V+E for adj. list

  - Graphs may not be connected

```
#define dfsR search
void dfsR(Graph G, Edge e)
  { int t, w = e.w;
    pre[w] = cnt++;
    for (t = 0; t < G->V; t++)
      if (G->adj[w][t] != 0)
        if (pre[t] == -1)
          dfsR(G, EDGE(w, t));
  }
```

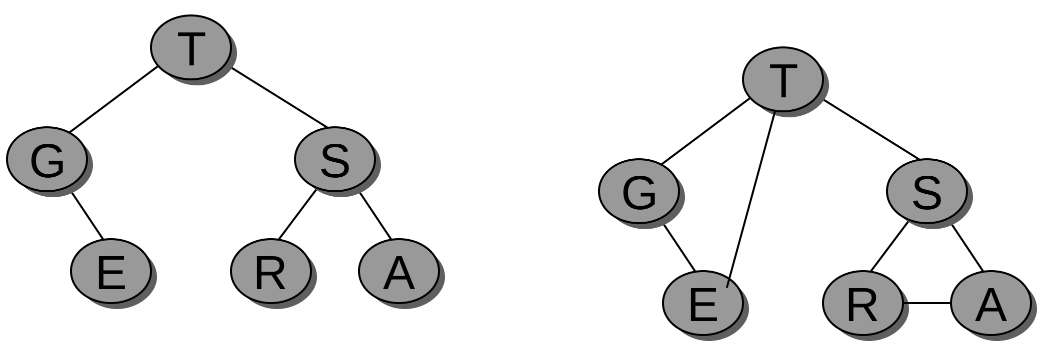
```
static int cnt, pre[maxV];
void GRAPHsearch(Graph G)
  { int v;
    cnt = 0;
    for (v = 0; v < G->V; v++) pre[v] = -1;
    for (v = 0; v < G->V; v++)
      if (pre[v] == -1)
        search(G, EDGE(v, v));
  }
```

DFS for graph problems

- Cycle detection
  - Back edges
- Simple path
- Simple connectivity
  - The graph search function calls the recursive DFS function only once.
- Two way Euler tour
  - Each edge visited exactly twice
- Spanning tree
  - Given a connected graph with V vertices, find a set of V-1 edges that connects the vertices
    - Any DFS is a spanning tree
- Two coloring, bipartiteness check

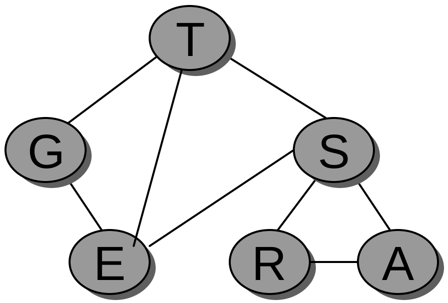
Separability and Connectivity

- Bridge
  - An edge that, if removed, would separate a connected graph into two disjoint subgraphs.
  - Edge-connected graph – has no bridges
    - In a DFS tree, edge v-w is a bridge iff there are no back edges that connect a descendant of w to an ancestor of w



Separability and Connectivity

- Articulation point (separation/cut)
  - Removal results in at least two disjoint subgraphs
- K-connected - for each pair:
  - At least k vertex disjoint paths
  - Indicates the number of vertices that need to be removed to disconnect a graph
  - Biconnected : 2-connected
    - removal of a vertex does not disconnect
- K-edge-connected - for each pair:
  - At least k edge disjoint paths
  - Indicates the number of edges that need to be removed to disconnect a graph



BFS Search

- Instead of Stack
  - Use a Queue
- Can be used to solve
  - Connected components
  - Spanning tree
  - Shortest paths

```
#define bfs search
void bfs(Graph G, Edge e)
  { int v, w;
    QUEUEput(e);
    while (!QUEUEempty())
      if (pre[(e = QUEUEget()).w] == -1)
        {
          pre[e.w] = cnt++; st[e.w] = e.v;
          for (v = 0; v < G->V; v++)
            if (G->adj[e.w][v] == 1)
              if (pre[v] == -1)
                QUEUEput(EDGE(e.w, v));
        }
  }
```

Directed Graph

- Digraph: Vertices + directed edges
  - In-degree: number of directed edge coming in
  - Out-degree: number of directed edge going out
  - DAG – no directed cycles
  - Strongly connected
    - Every vertex is reachable from every other
  - Not strongly connected : set of strong components
- Kernel K(D) of digraph D
  - One vertex of K(D) corresponds to each strong component of D
  - One edge in K(D) corresponds to each edge in D that connects vertices in different components
  - K(D) is a DAG

Reachability and Transitive closure

- Transitive closure of a graph
  - Same vertices + an edge from s to t in transitive closure if there is a directed path from s to t

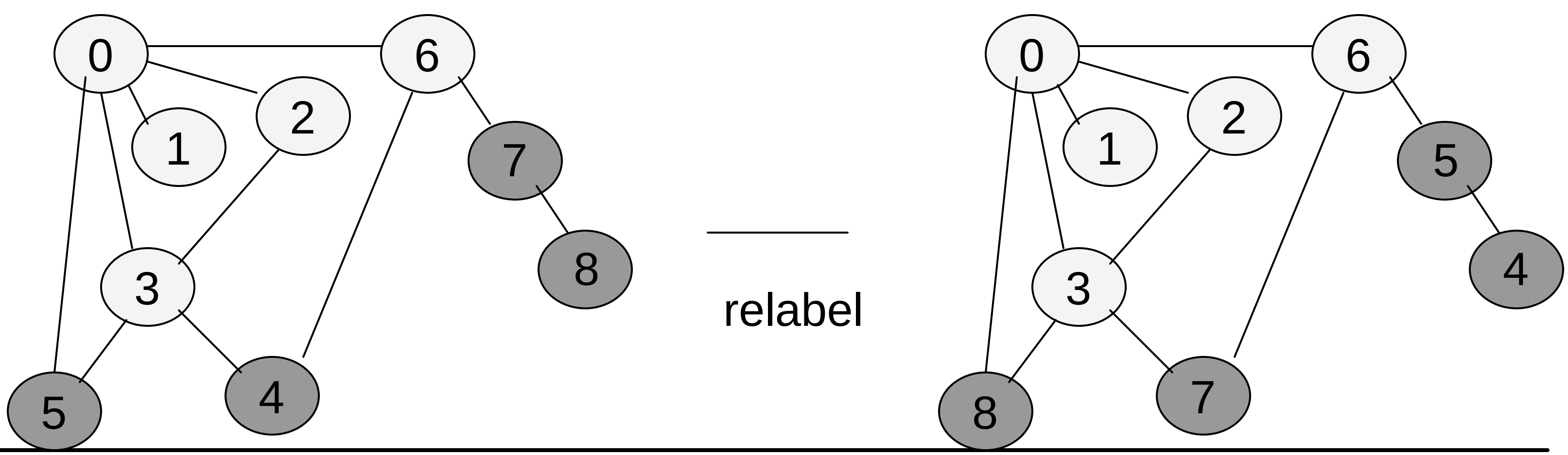
Warshall's algorithm
  - Complexity: V^3

```
for (i = 0; i < G->V; i++)
  for (s = 0; s < G->V; s++)
    for (t = 0; t < G->V; t++)
      if (A[s][i] && A[i][t] == 1) G->tc[s][t] = 1;
```

```
void GRAPHtc(Graph G)
 { int i, s, t;
   G->tc = MATRIXint(G->V, G->V, 0);
   for (s = 0; s < G->V; s++)
     for (t = 0; t < G->V; t++)
       G->tc[s][t] = G->adj[s][t];
   for (s = 0; s < G->V; s++) G->tc[s][s] = 1;
   for (i = 0; i < G->V; i++)
     for (s = 0; s < G->V; s++)
       if (G->tc[s][i] == 1)
         for (t = 0; t < G->V; t++)
           if (G->tc[i][t] == 1) G->tc[s][t] = 1;
 }
```

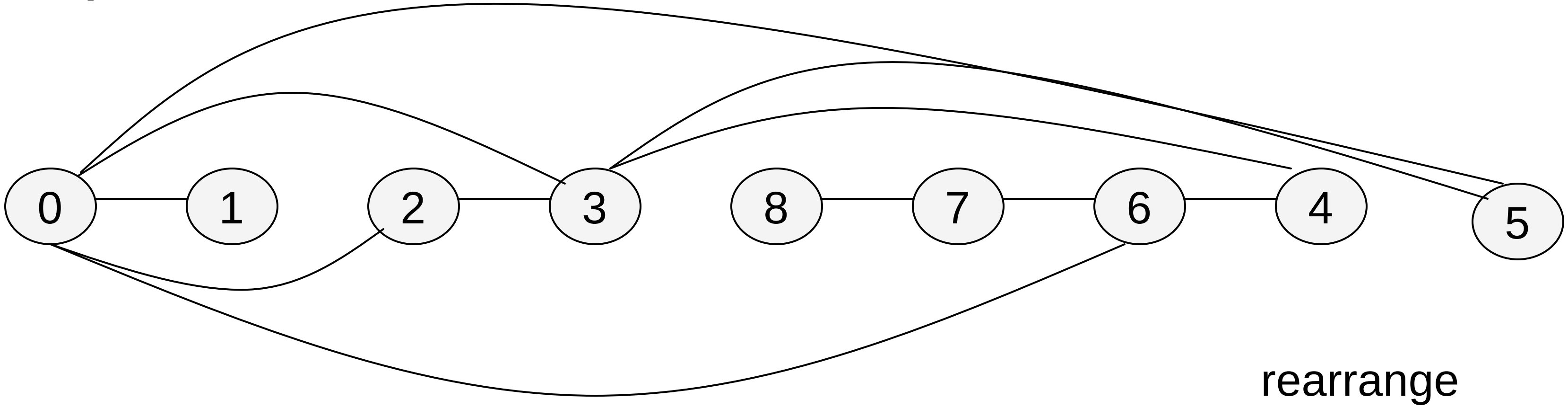
Topological Sort

- Given a DAG
  - Renumber vertices such that every directed edge points from a lower-numbered vertex to a higher-number one



Topological Sort

- Process each vertex before processing the vertices it points



rearrange

- Reverse topological sort
  - Scheduling applications
  - Postorder numbering in DFS yields a reverse topological sort

Topological Sort

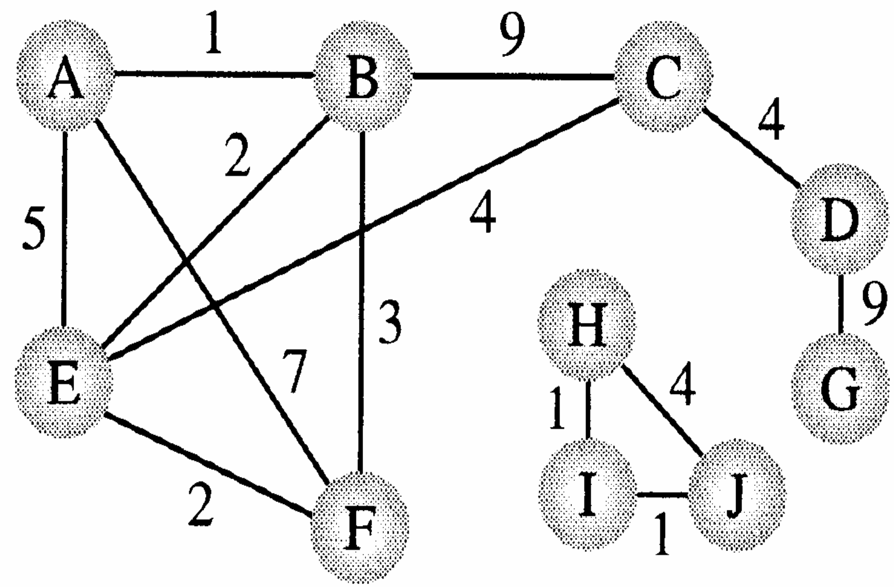
```
// Reverse (adj list)

static int cnt0;
static int pre[maxV];
void DAGts(Dag D, int ts[])
  { int v;
    cnt0 = 0;
    for (v = 0; v < D->V; v++)
      { ts[v] = -1; pre[v] = -1; }
    for (v = 0; v < D->V; v++)
      if (pre[v] == -1) TSdfsR(D, v, ts);
}
void TSdfsR(Dag D, int v, int ts[])
  { link t;
    pre[v] = 0;
    for (t = D->adj[v]; t != NULL; t = t->next)
      if (pre[t->v] == -1) TSdfsR(D, t->v, ts);
    ts[cnt0++] = v;
  }
```

```
// Adj. matrix
void TSdfsR(Dag D, int v, int ts[])
  { int w;
    pre[v] = 0;
    for (w = 0; w < D->V; w++)
      if (D->adj[w][v] != 0)
        if (pre[w] == -1) TSdfsR(D, w, ts);
    ts[cnt0++] = v;
  }
```

Minimum Spanning Tree

- Weighted graph
  - To incorporate this information into the graph, a weight, usually a positive integer, is attached to each arc
  - capacity, length, traversal time, or traversal cost.

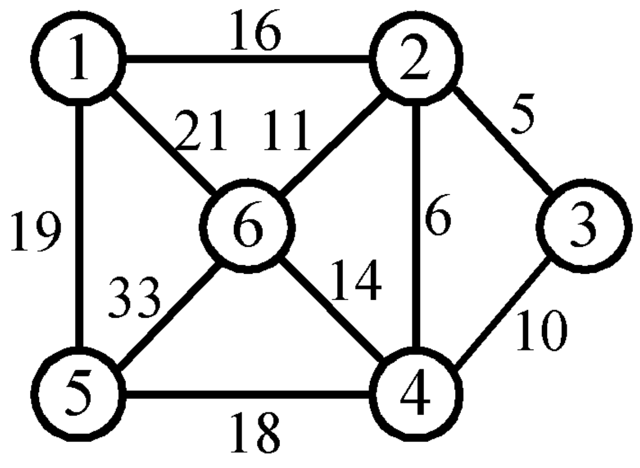


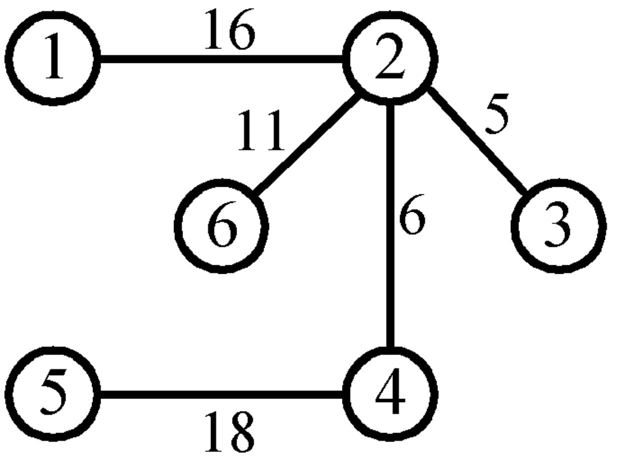
Minimum Spanning tree (MST)

- A spanning tree whose weight (the sum of the weights in its edges) is no larger than the weight of any other spanning tree
- Representation
  - weighted graph using an adjacency matrix is straightforward – use an integer matrix
  - In the adjacency list representation, the elements of the list now have two components, the node and the weight of the arc

MST

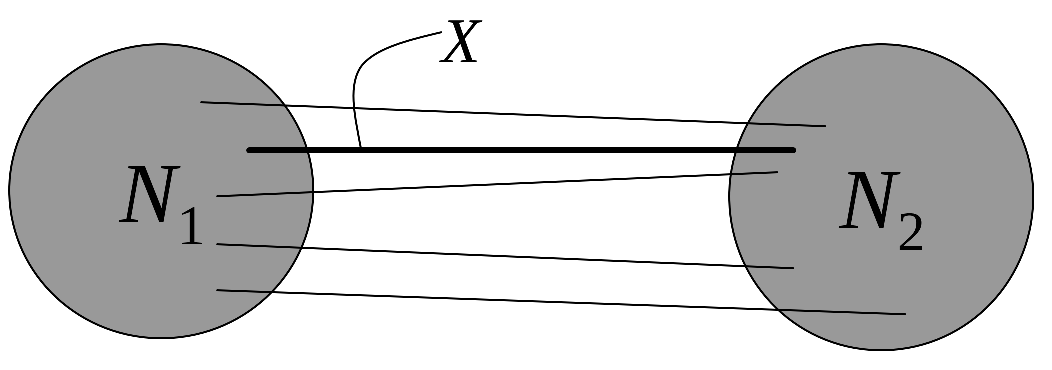
- A graph and its MST





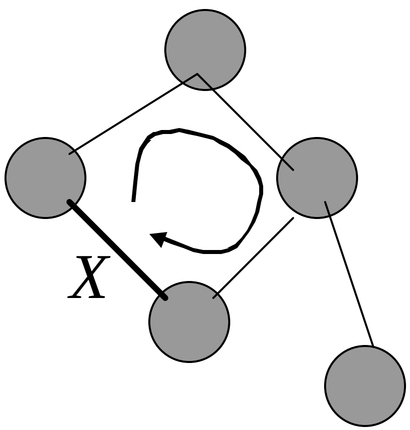
MST

- A Cut
  - A partition of the vertices into two disjoint sets
  - Crossing edge is one that connects a vertex in one set with a vertex in the other
- Cut Property
  - Given some cut in a graph, every minimal crossing edge belongs to some MST of the graph, and every MST contains a minimal crossing edge

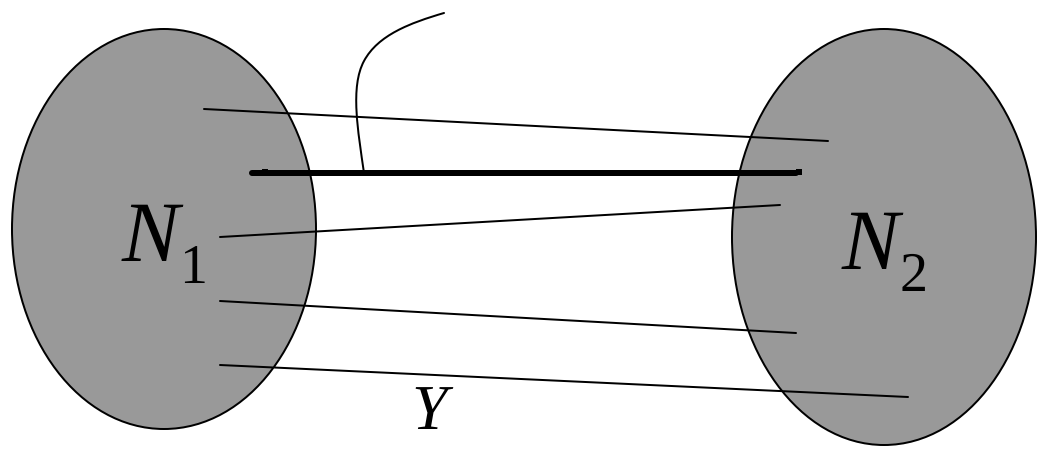


Cut Property

- **Proof**: Suppose that on the contrary, there is no minimum spanning tree that contains *X*. Take any minimum spanning tree and add the arc *X* to it.



A cycle is formed after adding *X*.



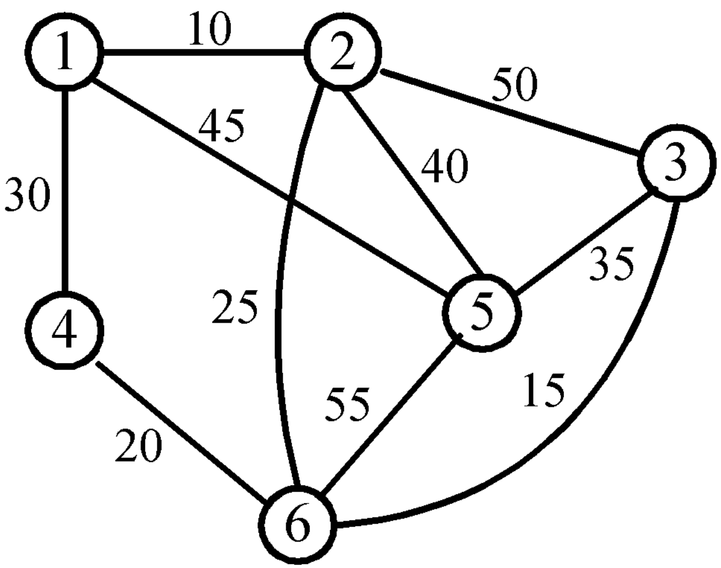
Cycle Property

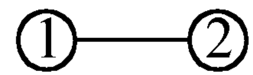
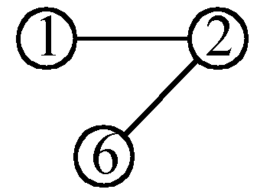
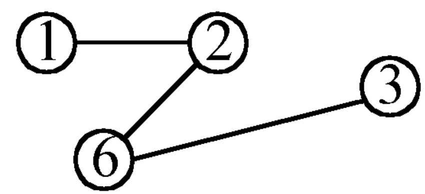
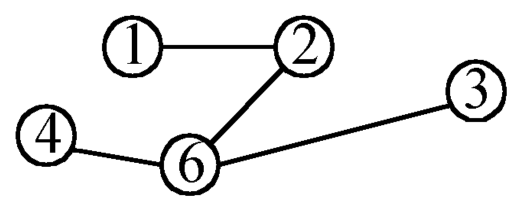
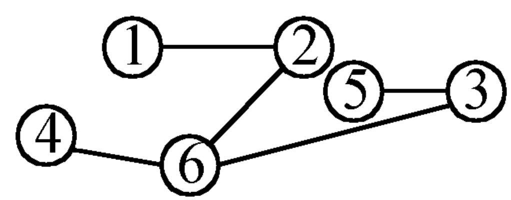
- Cycle property
  - Given a graph G, consider the graph G' defined by adding an edge e to G
  - Adding e to an MST of G and deleting a maximal edge on the resulting cycle gives an MST of G'

Prim's algorithm

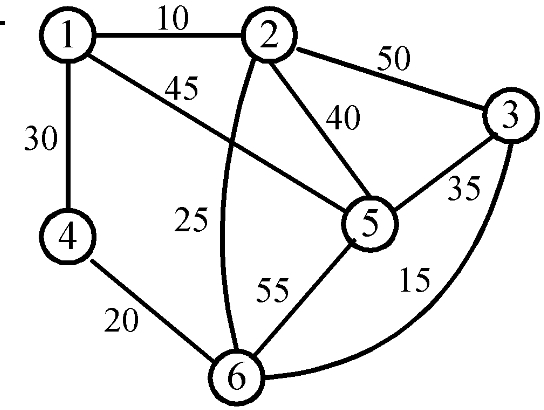
- **Prim's algorithm**
  - Step 1: $x \in V$, Let $A = \{x\}$, $B = V - \{x\}$
  - Step 2: Select $(u, v) \in E$, $u \in A$, $v \in B$ such that (u, v) has the smallest weight between A and B
  - Step 3: (u, v) is in the tree. $A = A \cup \{v\}$, $B = B - \{v\}$
  - Step 4: If $B = \emptyset$, stop; otherwise, go to Step 2.
- time complexity:
  - O(n2), $n = |V|$.

Prim's Algorithm

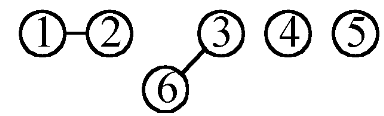
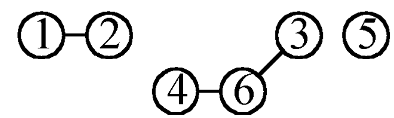
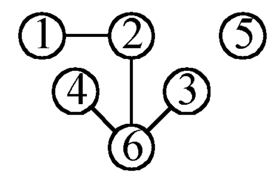
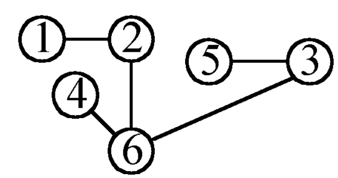


Edge	Cost	Spanning tree
(1,2)	10	
(2,6)	25	
(3,6)	15	
(6,4)	20	
(3,5)	35	

Kruskal's algorithm



- Step 1: Sort all edges
- Step 2: Add the next smallest weight edge to the forest if it will not cause a cycle.
- Step 3: Stop if we have n-1 edges. Otherwise, go to Step2.

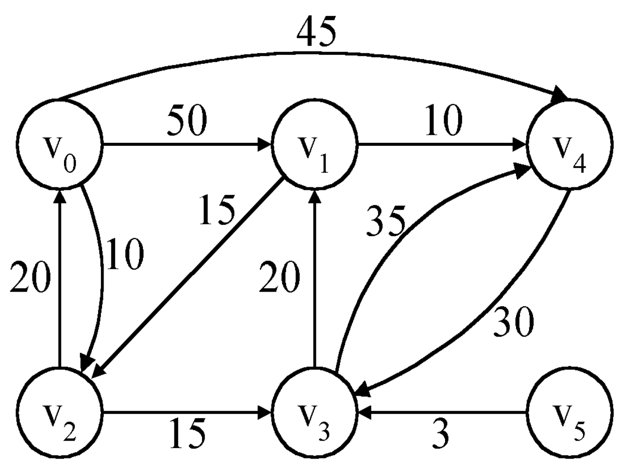
Edge	Cost	Spanning Forest
		
(1,2)	10	
(3,6)	15	
(4,6)	20	
(2,6)	25	
(1,4)	30	(reject)
(3,5)	35	

Shortest Path

- The shortest path problem has several different forms:
  - Given two nodes A and B, find the shortest path in the weighted graph from A to B.
  - Given a node A, find the shortest path from A to every other node in the graph. (**single-source shortest path problem**)
  - Find the shortest path between every pair of nodes in the graph. (**all-pair shortest path problem**)

Shortest Path

- Visit the nodes in order of their closeness;
  - visit A first, then visit the closest node to A,
  - then the next closest node to A, and so on.
- Dijkstra's algorithm



(a)

	Path	Length
1)	v_0v_2	10
2)	$v_0v_2v_3$	25
3)	$v_0v_2v_3v_1$	45
4)	v_0v_4	45

(b)

Shortest path

To select the next node to visit, we must choose the node in the fringe that has the shortest path to A. The shortest path from the next closest node must immediately go to a visited node.