IS 2610: Data Structures

Recursion, Divide and conquer

Dynamic programming,

Feb 2, 2004

Recursion and Trees

- Recursive algorithm is one that solves a problem by solving one or more smaller instances of the same problem
  - Functions that call themselves
  - Can only solve a base case Recursive function calls itself
- If not base case
  - Break problem into smaller problem(s)
  - Launch new copy of function to work on the smaller problem (recursive call/recursive step)
    - Slowly converges towards base case
    - Function makes call to itself inside the return statement
  - Eventually base case gets solved
  - Answer works way back up, solves entire problem

Algorithm for pre-fix expression

```
char *a; int i;
int eval()
  {   int x = 0;
      while (a[i] == ' ') i++;
      if (a[i] == '+')
         { i++; return eval() + eval(); }
      if (a[i] == '*')
         { i++; return eval() * eval(); }
      while ((a[i] >= '0') && (a[i] <= '9'))
          x = 10*x + (a[i++]-'0');
      return x;
  }
```

```
eval () * + 7 6 12
    eval() + 7 6
        eval () 7
        eval () 6
        return 13 = 7 + 6
    eval () 12
    return 12 * 13
```

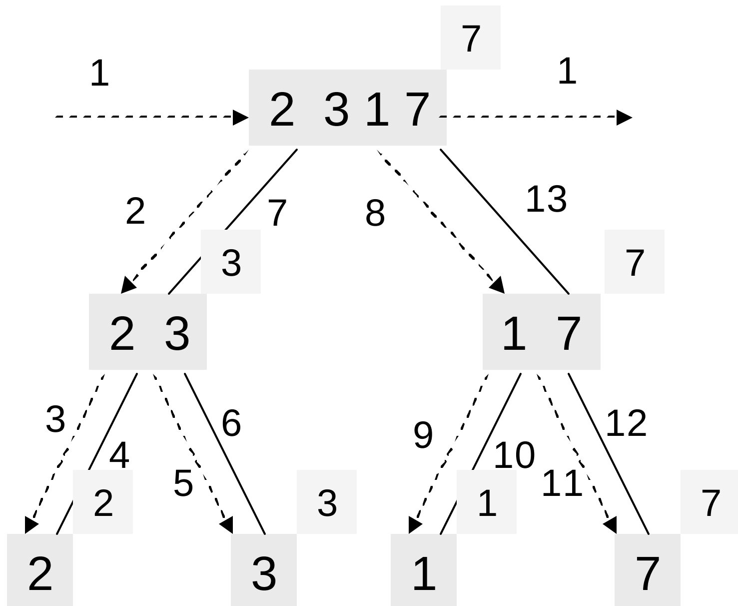
Recursive vs. iterative solution

- In principle, a loop can be replaced by an equivalent recursive program
  - Recursive program usually is more natural way to express computation
- Disadvantage
  - Nested function calls –
    - Use built in pushdown stack
    - Depth will depend on input
    - Hence programming environment has to maintain a stack that is proportional to the push down stack
    - Space complexity could be high

Divide and Conquer

- Many recursive programs use recursive calls on two subsets of inputs (two halves usually)
  - Divide the problem and solve them – divide and conquer paradigm
  - Property 5.1: a recursive function that divides a problem size N intro two independent (nonempty) parts that it solves recursively calls itself less than N times
  - Complexity: $T_N = T_k + T_{N-k} + 1$

Find max- Divide and Conquer




```
Item max(Item a[], int l, int r)
{  Item u, v;
   int m = (l+r)/2;
   if (l == r) return a[l];
   u = max(a, l, m);
   v = max(a, m+1, r);
   if (u > v) return u;
   else return v;
}
```

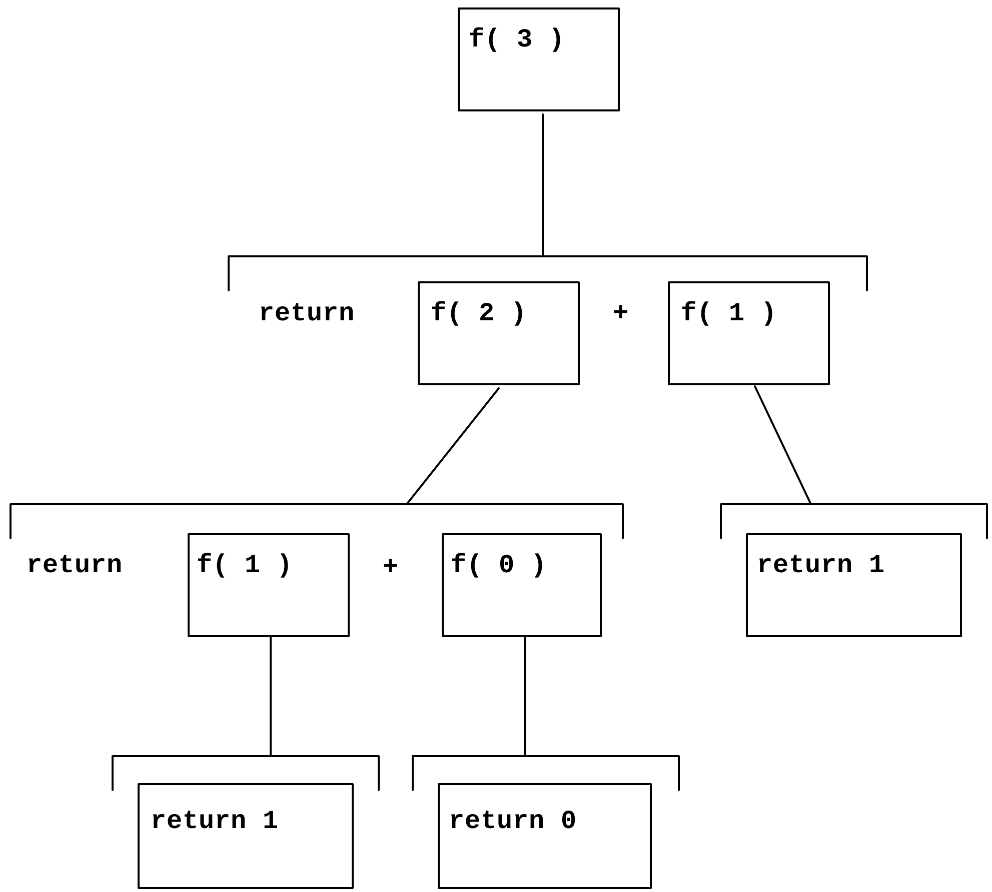
Dynamic programming

- When the sub-problems are not independent the situation may be complicated
  - Time complexity can be very high
- Example
  - Fibonacci number
    - Base case: $F_0 = F_1 = 1$
    - $F_n = F_{n\text{-}1} + F_{n\text{-}2}$

```
int fibanacci(int n){
    if (n=<1) return 1; // Base case
    return fibonacci(n-1) + fibonacci(n-2);
}
```

Recursion: Fibonacci Series

- Order of operations
  - `return fibonacci( n - 1 ) + fibonacci( n - 2 );`
- Recursive function calls
  - Each level of recursion doubles the number of function calls
    - 30th number = 2^30 ~ 4 billion function calls
  - Exponential complexity



Simpler Solution

- Linear!!
- Observation
  - We can evaluate any function by computing all the function values in order starting at the smallest, using previously computed values at each step to compute the current value
    - Bottom-up Dynamic programming
      - Applies to any recursive computation, *provided* that we can afford to save all the previously computed values
  - Top-down
    - Modify the recursive function to save the computed values and to allow checking these saved values
      - Memoization

```
F[0] = F[1] = 1;
For (i = 2; i<=N; i++);
    F[0] = F[i-1] + F[i-2];
```

Dynamic Programming

- Top-down : save known values
- Bottom-up : pre-compute values
  - Determining the order may be a challenge
- Top-down preferable
  - It is a mechanical transformation of a natural problem
  - The order of computing the sub-problems takes care of itself
  - We may not need to compute answers to all the sub-problems

```
int F(int i)
{   int t;
    if (knownF[i] != unknown)
        return knownF[i];
    if (i == 0) t = 0;
    if (i == 1) t = 1;
    if (i > 1) t = F(i-1) + F(i-2);
       return knownF[i] = t;
}
```

Dynamic programming
Knapsack problem

- *Property*: DP reduces the running times of a recursive function to be at most the time required to evaluate the function for all arguments less than or equal to the given argument
- Knapsack problem
  - Given
    - N types of items of varying size and value
    - One knapsack (belongs to a thief!)
  - Find: the combination of items that maximize the total value

Knapsack problem

Knapsack size: 17

	0	1	2	3	4
Item	A	B	C	D	E
Size	3	4	7	8	9
Val	4	5	10	11	13

```
int knap(int cap)
{  int i, space, max, t;
   for (i = 0, max = 0; i < N; i++)
       if ((space = cap - items[i].size) >= 0)
          if ((t = knap(space) + items[i].val) > max)
             max = t;
    return max;
}
```

```
int knap(int M)
{  int i, space, max, maxi, t;
   if (maxKnown[M] != unknown) return maxKnown[M];
   for (i = 0, max = 0; i < N; i++)
        if ((space = M-items[i].size) >= 0)
            if ((t = knap(space) + items[i].val) > max)  { max = t; maxi = i; }
   maxKnown[M] = max; itemKnown[M] = items[maxi];
return max; }
```

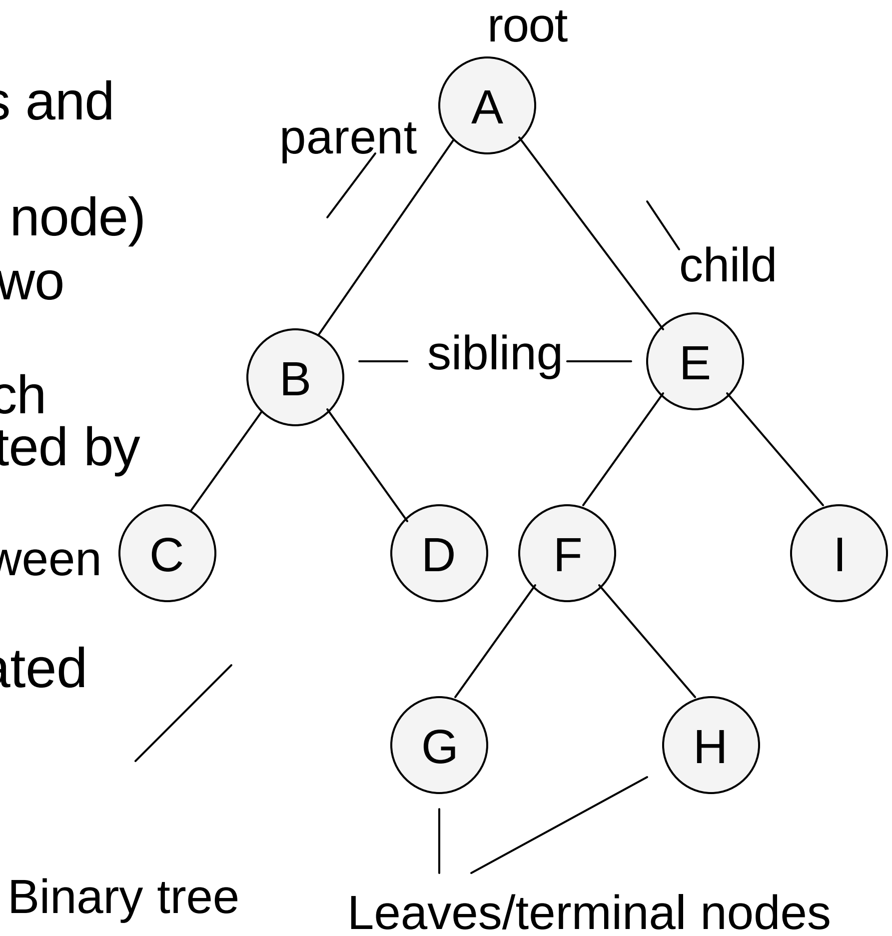
Tree

- Trees are central to design and analysis of algorithms
  - Trees can be used to describe dynamic properties
  - We build and use explicit data structures that are concrete realization of trees

  General issues:
  - Trees
  - Rooted tree
  - Ordered trees
  - M-ary trees and binary trees

Tree

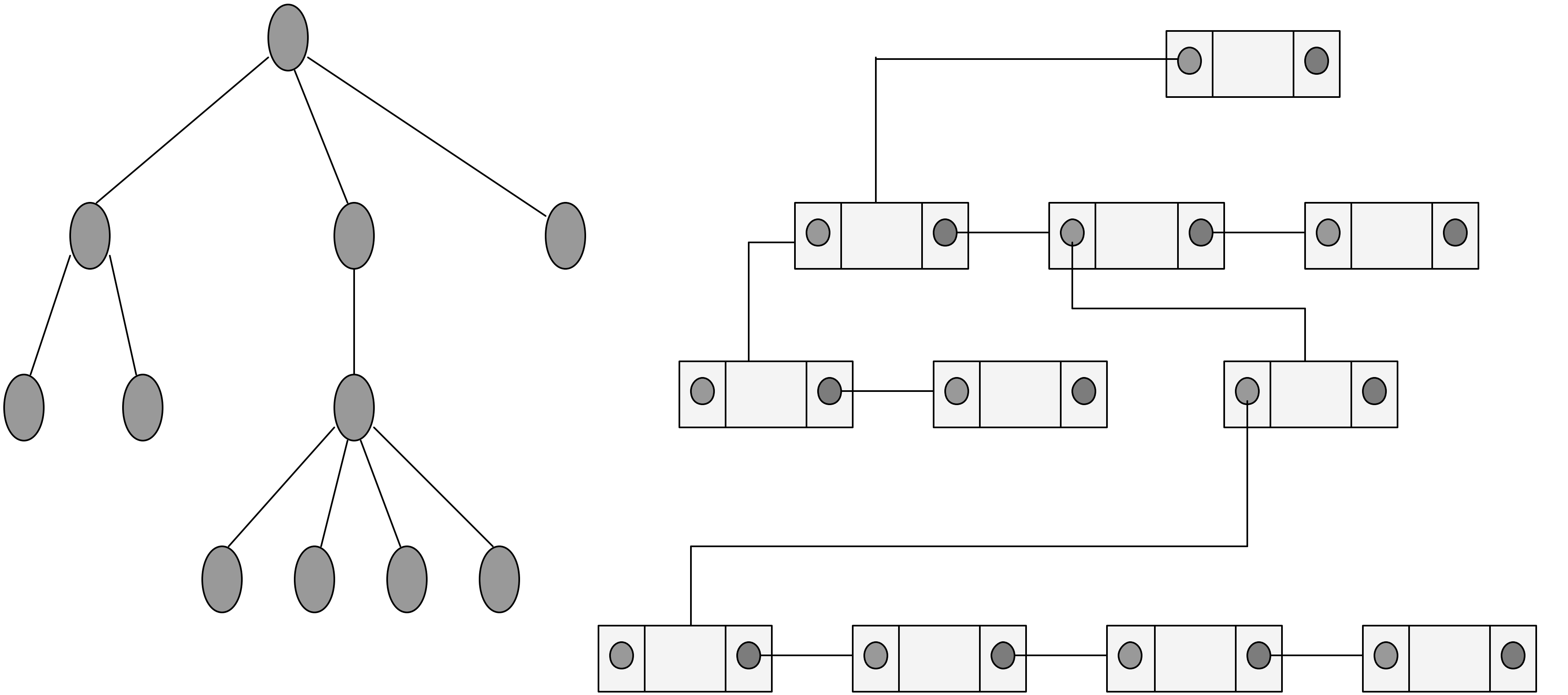
- Trees
  - Non-empty collection of vertices and edges
  - Vertex is a simple object (a.k.a. node)
  - Edge is a connection between two nodes
  - Path is a distinct vertices in which successive vertices are connected by edges
    - There is precisely one path between any two vertices
- Rooted tree: one node is designated as the root
- Forest
  - Disjoint set of trees



Definitions

- Binary tree is either an external node or an internal node connected to a pair of binary trees, which are called the left sub-tree and the right sub-tree of that node
  - Struct node {Item item; link left, link right;}
- M-ary tree is either an external node or an internal node connected to an ordered sequence of M-trees that are also M-ary trees
- A tree (or ordered tree) is a node (called the root) connected to a set of disjoint trees. Such a sequence is called a forest.
  - Arbitrary number of children
    - One for linked list connecting to its sibling
    - Other for connecting it to the sibling

Example general tree



Binary trees

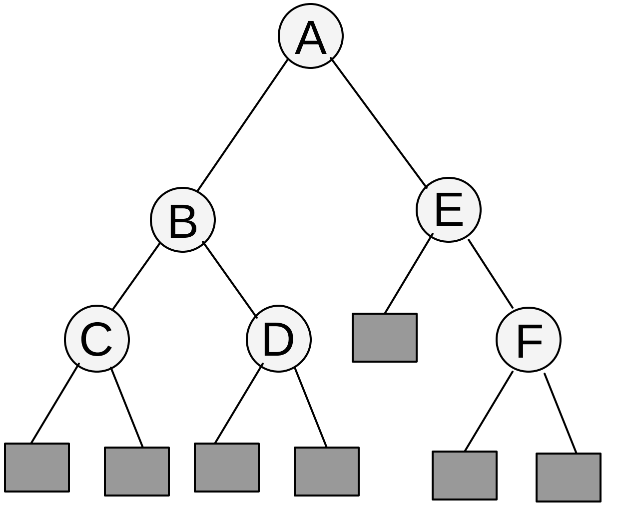
- A binary tree with N internal nodes has N+ 1 external nodes
  - Proof by induction
  - $N = 0$ (no internal nodes) has one external node
  - Hypothesis: holds for N-1
  - k, N -1 - k internal nodes in left and right sub-trees (for k between 0 and N-1)
  - $(k+1) + (N - 1 - k) = N + 1$

Binary tree

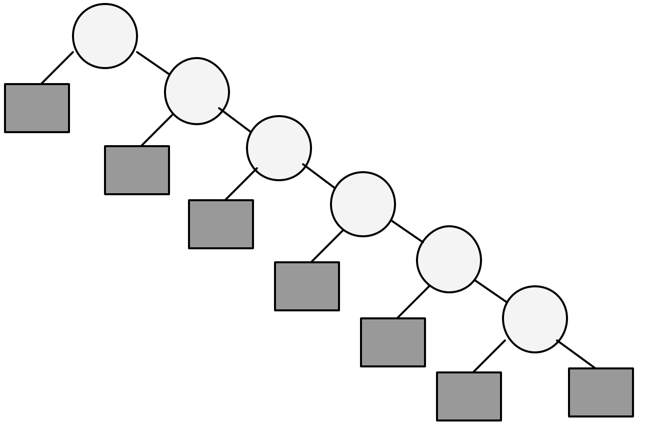
- A binary tree with N internal nodes has 2N links
  - N-1 to internal nodes
    - Each internal node except root has a unique parent
    - Every edge connects to its parent
  - N+1 to external nodes
- Level, height, path
  - Level of a node is 1 + level of parent (Root is at level 0)
  - Height is the maximum of the levels of the tree’s nodes
  - Path length is the sum of the levels of all the tree’s nodes
  - Internal path length is the sum of the levels of all the internal nodes

Examples

- Level of D ?
- Height of tree?
- Internal length?
- External length?



- Height of tree?
- Internal length?
- External length?



Binary Tree

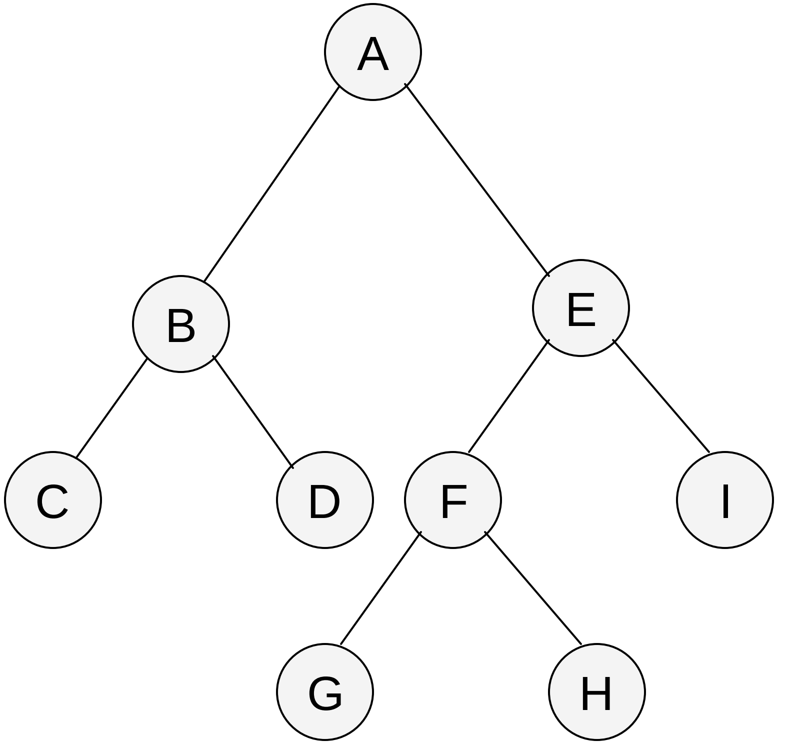
- External path length of any binary tree with N internodes is $2N$ greater than the internal path length
- The height of a binary tree with N internal nodes is at least $\lg N$ and at most N-1
  - Worst case is a degenerate tree: N-1
  - Best case: balanced tree with 2^i nodes at level i.
    - Hence for height: $2^{h-1} < N+1 = 2^h$ – hence h is the heigth

Binary Tree

- Internal path length of a binary tree with N internal nodes is at least $N\ lg\ (N/4)$ and at most $N(N\text{-}1)/2$
  - Worst case : $N(N\text{-}1)/2$
  - Best case: $(N+1)$ external nodes at height no more than $\lfloor lg\ N \rfloor$
    - $(N+1)\lfloor lg\ N \rfloor - 2N < N \lg (N/4)$

Tree traversal (binary tree)

- Preorder
  - Visit a node,
  - Visit left subtree,
  - Visit right subtree
- Inorder
  - Visit left subtree,
  - Visit a node,
  - Visit right subtree
- Postorder
  - Visit left subtree,
  - Visit right subtree
  - Visit a node



Recursive / Nonrecursive Preorder

```
void traverse(link h, void (*visit)(link))
{
   If (h == NULL) return;
   (*visit)(h);
   traverse(h->l, visit);
   traverse(h->r, visit);
}
```

```
void traverse(link h, void (*visit)(link))
{
   STACKinit(max);
   STACKpush(h);
  while (!STACKempty())
  {
        (*visit)(h = STACKpop());
        if (h->r != NULL) STACKpush(h->r);
       if (h->l != NULL) STACKpush(h->l);
  }
}
```

Recursive binary tree algorithms

- Exercise on recursive algorithms:
  - Counting nodes
  - Finding height

Sorting Algorithms

- Selection sort
  - Find smallest element and put in the first place
  - Find next smallest and put in second place
  - ..
  - Try out ! Complexity ? Recursive?
- Bubble sort
  - Move through the elements exchanging adjacent pairs if the first one is larger than the second
  - Try out ! Complexity ?

Insertion sort

- “People” method

2 6 3 1 5

```
#define less(A, B) (key(A) < key(B))
#define exch(A, B) { Item t = A; A = B; B = t; }
#define compexch(A, B) if (less(B, A)) exch(A, B)

void insertion(Item a[], int l, int r) {
   int i;
   for (i = l+1; i <= r; i++)
        compexch(a[l], a[i]);
   for (i = l+2; i <= r; i++)   {
      int j = i; Item v = a[i];
      while (less(v, a[j-1])) {
          a[j] = a[j-1]; j--;
      }
      a[j] = v;
   }
}
```