IS 2610: Data Structures

Graph

April 12, 2004

Graph

- Weighted graph – call it networks
- Shortest path between nodes s and t in a network
  - Directed simple path from s to t with the property that no other such path has a lower weight
  - Negative edges?
- Applications ?

Shortest Path

- The shortest path problem has several different forms:
  - Source-sink SP:
    - Given two nodes A and B, find the shortest path in the weighted graph from A to B.
  - Single source SP:
    - Given a node A, find the shortest path from A to every other node in the graph.
  - All Pair SP:
    - Find the shortest path between every pair of nodes in the graph

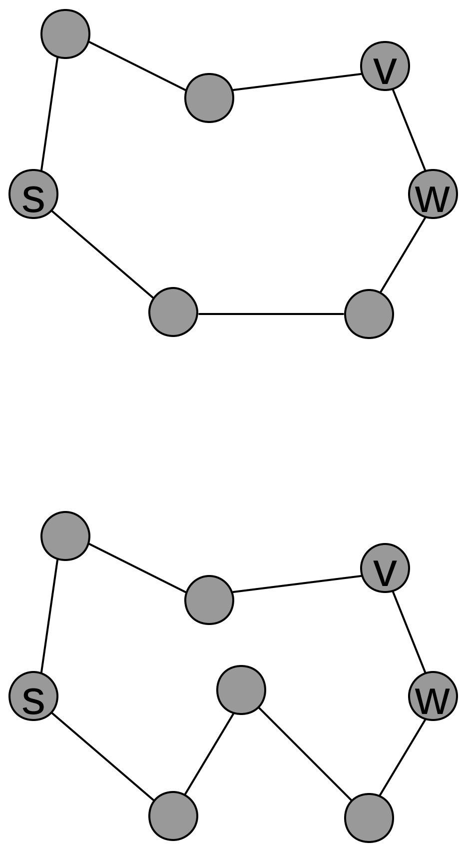
Basic concept in SP

- Relaxation
  - At each step increment the SP information
  - Path relaxation
    - Test if traveling through a given vertex introduces a new shortest path between a pair of vertices
  - Edge relaxation : special case of path relaxation
    - Test if traveling through a given edge gives a new shortest path to its destination vertex

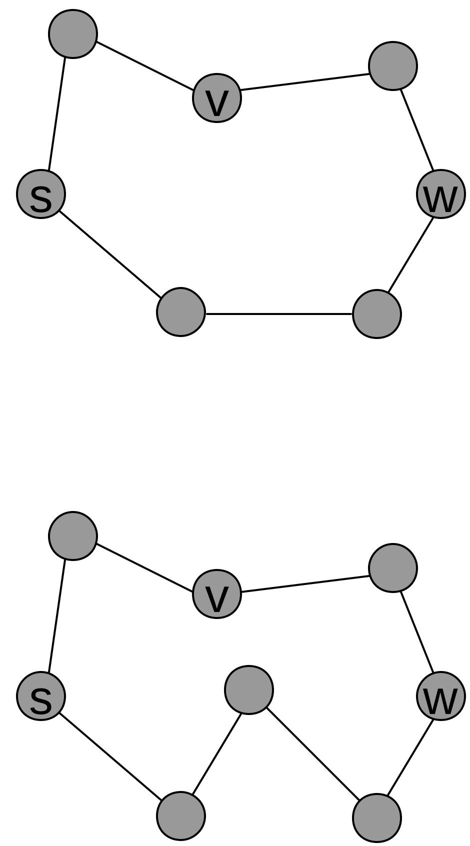
      If (wt[w] > wt[v] + e.wt)

      { wt[w] = wt[v] + e.wt; st[w] = v;}

Relaxation



Edge relaxation



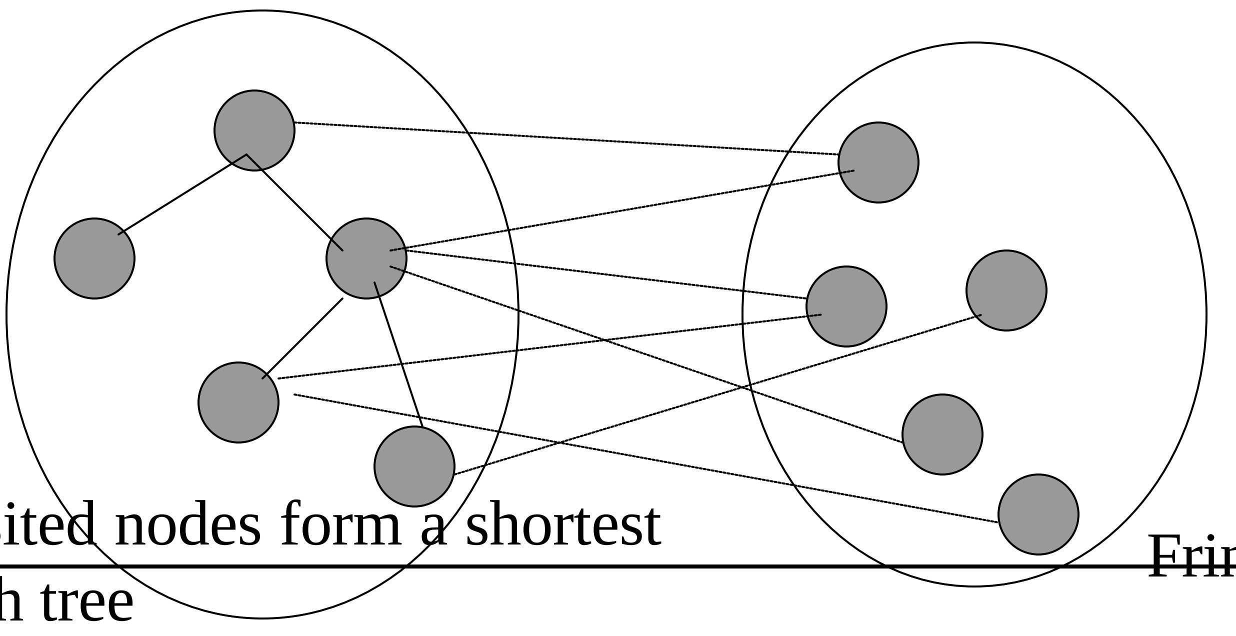
Path relaxation

Shortest Path

- Property 21.1
  - If a vertex x is on a shortest path from s to t, then that path consists of a shortest path from s to x followed by a shortest path from x to t
- Dijkstra’s algorithm (similar to Prim’s MST)
  - Start at source
  - Include next edge that gives the shortest path from the source to a vertex not in the SP

Shortest path

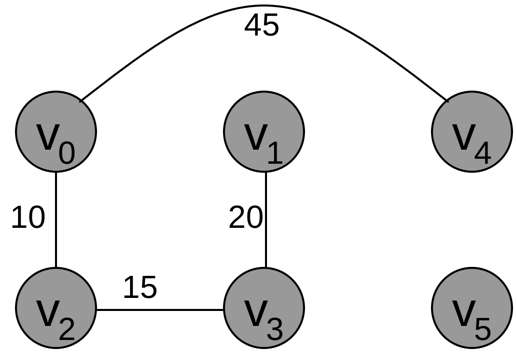
To select the next node to visit, we must choose the node in the fringe that has the shortest path to A. The shortest path from the next closest node must immediately go to a visited node.



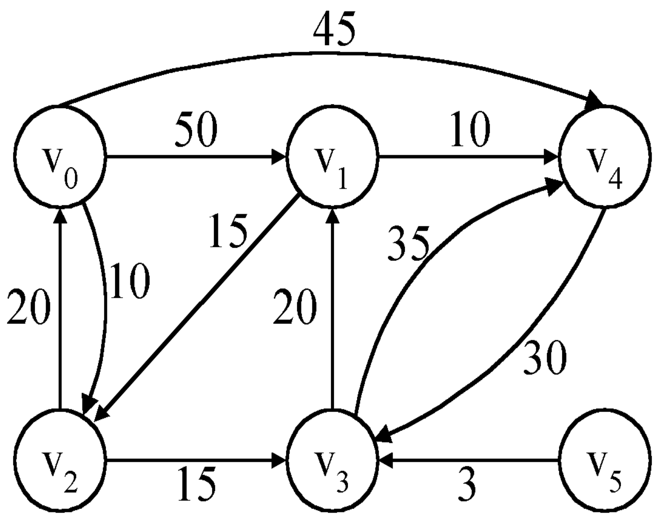
Visited nodes form a shortest path tree

Fringe node set

Dijkstra's algorithm



- Complexity?



(a)

	Path	Length	Fringe
1)	v_0v_2	10	v_0v_1, v_0v_2, v_0v_4
2)	$v_0v_2v_3$	25	v_0v_1, v_0v_4, v_2v_3
3)	$v_0v_2v_3v_1$	45	v_0v_1, v_0v_4, v_3v_1 (v_3v_4?)
4)	v_0v_4	45	v_0v_4

(b)

Dijkstra’s algorithm

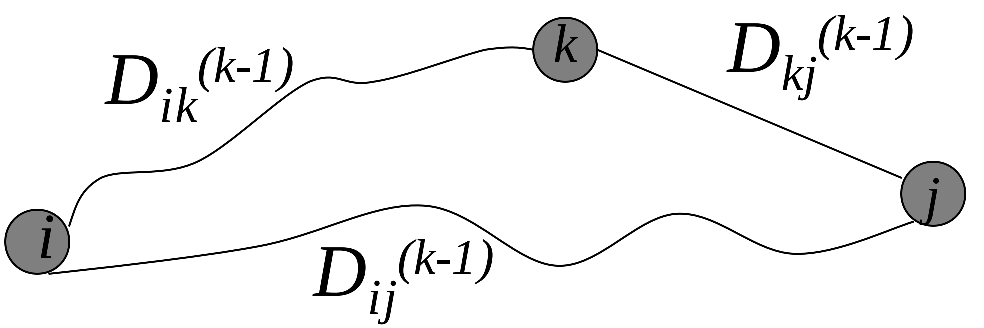
```
#define GRAPHpfs GRAPHspt
#define P (wt[v] + t->wt)
void GRAPHpfs(Graph G, int s, int st[], double
wt[])
  { int v, w;  link t;
    PQinit(); priority = wt;
    for (v = 0; v < G->V; v++)
      { st[v] = -1; wt[v] = maxWT; PQinsert(v); }
    wt[s] = 0.0; PQdec(s);
    while (!PQempty())
      if (wt[v = PQdelmin()] != maxWT)
        for (t = G->adj[v]; t != NULL; t = t->next)
          if (P < wt[w = t->v])
            { wt[w] = P; PQdec(w); st[w] = v; }
  }
```

All-pairs shortest path

- Use Dijkstra's algorithm from each vertex
  - Complexity: $VE\lg V$
- Floyd's algorithm
  - Use extension of Warshall's algorithm for transitive closure
  - Complexity: V^3

Floyd-Warshall Algorithm

- $D_{ij}^{(k)}$ = length of shortest path from i to j with intermediate vertices from $\{1, 2, ..., k\}$:
- Dynamic Programming: recurrence
  - $D_{ij}^{(0)} = D_{ij}$
  - $D_{ij}^{(k)} = \min \{D_{ij}^{(k-1)}, D_{ik}^{(k-1)} + D_{kj}^{(k-1)}\}$



intermediate nodes in $\{1, 2, ..., k\}$

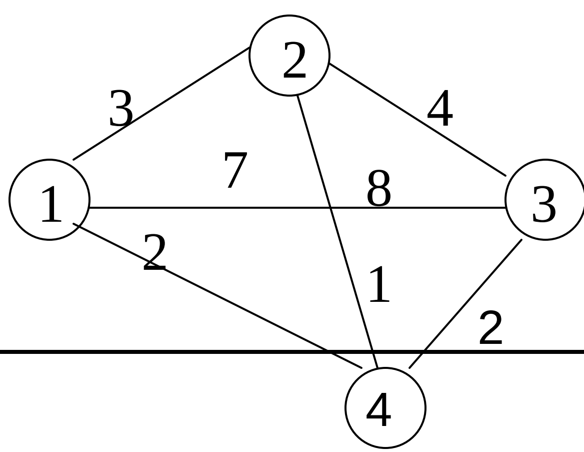
The Floyd-Warshall algorithm

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 \end{cases}$$

Floyd-Warshall(W) $\Theta(n^3)$

```
n ← rows[W]
D(0) = W
for k ← 1 to n
        do for i ← 1 to n
           do for j ← 1 to n
                  d_ij(k) ← min(d_ij(k-1), d_ik(k-1) + d_kj(k-1))
return D(n)
```

$$d_{ij}^{(k)} \leftarrow \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$



calculate $D^{(0)}$, $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, $D^{(4)}$ and $D^{(5)}$

Floyd-Warshall Algorithm

```
void GRAPHspALL(Graph G)
  { int i, s, t;
    double **d = MATRIXdouble(G->V, G->V,
maxWT);
    int **p = MATRIXint(G->V, G->V, G->V);
    for (s = 0; s < G->V; s++)
      for (t = 0; t < G->V; t++)
        if ((d[s][t] = G->adj[s][t]) < maxWT)
          p[s][t] = t;
    for (i = 0; i < G->V; i++)
      for (s = 0; s < G->V; s++)
        if (d[s][i] < maxWT)
          for (t = 0; t < G->V; t++)
            if (d[s][t] > d[s][i]+d[i][t])
              { p[s][t] = p[s][i];
                d[s][t] = d[s][i]+d[i][t]; }
    G->dist = d; G->path = p;
  }
```

Complexity classes

- An algorithm A is of polynomial complexity if there exists a polynomial $p(n)$ such that the computing time of A is $O(p(n))$
- Set P
  - set of decision problems solvable in polynomial time using deterministic algorithm
    - **Deterministic**: result of each step is uniquely defined
- Set NP
  - set of decision problem solvable in polynomial time using nondeterministic algorithm
    - **Non-deterministic**: result of each step is a set of possibilities
- $P \subseteq NP$
  - Problem is $P = NP$ or $P \neq NP$?

Complexity classes

- Satisfiability is in P iff P = NP
- NP-hard problems
  - Reduction L1 reduces to L2
    - Iff there is a way to solve L1 in deterministic polynomial time algorithm using deterministic algorithm that solves L2 in polynomial time
  - A problem L is NP-hard iff satisfiability reduces to L
- NP-complete
  - L is in NP
  - L is NP-hard

Showing a problem NP-complete

- Show that it is in NP
- Show that it is NP-hard
  - Pick problem L already known to be NP-hard
  - Show that the problem can be reduced to L
- Example
  - Show that traveling salesman problem is NP-complete
  - Known: directed Hamiltonian cycle problem is NP-complete